

QUADRATIC AND PINCZON ALGEBRAS

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ABSTRACT. Given a symmetric non degenerated bilinear form b on a vector space V , G. Pinczon and R. Ushirobira defined a bracket $\{ , \}$ on the space of multilinear skewsymmetric forms on V . With this bracket, the quadratic Lie algebra structure equation on (V, b) becomes simply $\{\Omega, \Omega\} = 0$.

We characterize similarly quadratic associative, commutative or pre-Lie structures on (V, b) by the same equation $\{\Omega, \Omega\} = 0$, but on different spaces of forms. These definitions extend to quadratic up to homotopy algebras and allows to describe the corresponding cohomologies.

1. INTRODUCTION

In [PU, DPU] (see also [D, MPU]), Georges Pinczon and Rosane Ushirobira introduced what they called a Poisson bracket on the space of skewsymmetric forms on a finite dimensional vector space V , equipped with a symmetric, non degenerated, bilinear form b . If (e_i) is a basis in V and (e'_j) the basis defined by the relations $b(e'_j, e_i) = \delta_{ij}$, the bracket is:

$$\{\alpha, \beta\} = \sum_i \iota_{e_i} \alpha \wedge \iota_{e'_i} \beta.$$

Especially, if α is a $(k+1)$ -form, and β a $(k'+1)$, then $\{\alpha, \beta\}$ is a $(k+k')$ -form. Shifting the degree on V by -1, we replace V by $V[1]$, the skew-symmetric bilinear forms on V becomes symmetric forms on $V[1]$ and the bracket $\{ , \}$ a Lie bracket.

In fact, the authors proved that a structure of quadratic Lie algebra $(V, [,], b)$ on V is completely characterized by a 3-form I , such that $\{I, I\} = 0$. The relation between the Lie bracket and I is simply $I(x, y, z) = b([x, y], z)$, and the equation $\{I, I\} = 0$ is the structure equation. A direct consequence of this construction is the existence of a cohomology on the space of forms, given by:

$$d\alpha = \{\alpha, I\}.$$

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This cohomology characterizes the problem of definition and deformation of quadratic Lie algebra structure on (V, b) .

In this paper, we first generalize this construction, defining the Pinczon bracket on the space $\mathcal{C}(V)$ of cyclic multilinear forms on the shifted space $V[1]$. A Pinczon bracket on $\mathcal{C}(V)$ is a Lie bracket $(\Omega, \Omega') \mapsto \{\Omega, \Omega'\}$ such that $\{\Omega, \Omega'\}$ is $(k+k')$ -linear, if Ω is $(k+1)$ -linear, and Ω' $(k'+1)$ -linear, and, for each linear form α , $\{\alpha, \cdot\}$ is a derivation of the cyclic product.

As in [PU], there is a one-to-one correspondence between the set of Pinczon brackets on $\mathcal{C}(V)$ and the set of non degenerated, bilinear symmetric forms b on V , and the correspondence is given through the Pinczon-Ushirobira formula:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

On the other hand, it is well known that the space $\bigotimes^+ V[1]$ is a cogeбра for the comultiplication given by the deconcatenation map:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{i=1}^{k-1} (x_1 \otimes \dots \otimes x_i) \bigotimes (x_{i+1} \otimes \dots \otimes x_k).$$

Moreover, the coderivations of Δ are characterized by their Taylor series (Q_k) where $Q_k : \bigotimes^k V[1] \longrightarrow V[1]$, and the bracket $[Q, Q']$ of two such coderivation is still a coderivation.

Suppose now there is a symmetric non degenerated form b on V . Denotes B the corresponding form, but on $V[1]$. There is a bijective map between $\mathcal{C}(V)$ and the space \mathcal{D}_B of B -quadratic coderivation Q , given by the formula:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

This map is an isomorphism of Lie algebra: $\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']}$.

With this construction, the notion of quadratic associative algebra, respectively quadratic associative algebra up to homotopy does coincide with the notion of Pinczon algebra structure on $\mathcal{C}(V)$. This gives also an explicit way to refine the Hochschild cohomology defined by the algebra structure.

Moreover, the subspace $\mathcal{C}_{vsp}(V)$ of cyclic, vanishing on shuffle products forms is a Lie subalgebra of $\mathcal{C}(V)$. The restriction to this subalgebra of the above construction gives us the notion of quadratic commutative algebra (up to homotopy): it is a Pinczon algebra structure on $\mathcal{C}_{vsp}(V)$. Similarly, one refines the Harrison cohomology associated to commutative algebras.

A natural quotient of $(\mathcal{C}(V), \{ , \})$ is the Lie algebra $(\mathcal{S}, \{ , \})$ of totally symmetric multilinear forms on $V[1]$. This allows us to refine the notion of quadratic Lie algebra (up to homotopy), and the corresponding Chevalley cohomology.

Considering now bi-symmetric multilinear forms on $V[1]$: i.e. separately symmetric in their two last variables and in all their other variables, and extending canonically the Pinczon bracket to a bracket $\{ , \}^+$ on this space of forms, we can define similarly quadratic pre-Lie algebra structures on (V, b) , and the corresponding pre-Lie cohomology.

Finally, we study the natural example of the space of $n \times n$ matrices. An unpublished preprint ([A]) contains a part of these results.

2. CYCLIC FORMS

2.1. Koszul's rule.

In this paper, V is a finite dimensional graded vector space, on a characteristic 0 field \mathbb{K} . Denote $|x|$ the degree of a vector x in V .

First recall the sign rule due to Koszul (see [Ko]). For any relation between quantities using letters representing homogeneous objects, in different orderings, it is always understood that there is an implicit $+$ sign in front of the first term. For each other term, if σ is the permutation of the letters between the first quantity and the considered term, there is the implicit sign $\varepsilon_{|letters|}(\sigma)$ (the sign of σ , taking into account only positions of the odd degree letters) in front of it.

As usual, $V[1]$ is the space V with a shifted degree. If x is homogeneous, its degree in $V[1]$ is $\deg(x) = |x| - 1$. Note simply x for $\deg(x)$ when no confusion is possible. Very generally, we use a small letter for each mapping defined on V , and capital letter for the ‘corresponding’ mapping, defined on $V[1]$. Let us define now these corresponding mappings.

For any k , define a ‘ k -cocycle’ η by putting $\eta(x_1, \dots, x_k) = (-1)^{\sum_{j \leq k} (k-j)x_j}$. Then, if $\varepsilon(\sigma)$ is the sign of the permutation σ in \mathfrak{S}_k ,

$$\eta(x_{\sigma(1)}, \dots, x_{\sigma(k)})\eta(x_1, \dots, x_k) = \varepsilon(\sigma)\varepsilon_{|x|}(\sigma)\varepsilon_x(\sigma).$$

If q is a k -linear mapping from V^k into a graded vector space W , define the associated k -linear mapping from $V[1]^k$ into $W[1]$, with $\deg(Q) = |q| + k - 1$, by

$$Q(x_1, \dots, x_k) = \eta(x_1, \dots, x_k)q(x_1, \dots, x_k).$$

Preferring to keep the 0 degree for scalars, if ω is a k -linear form on V , the same formula associates to ω a form Ω , on $V[1]$, with degree $\deg(\Omega) = |\omega| + k$.

This shift of degree modifies the symmetry properties of these mappings. For instance, if q is σ -invariant, then Q is σ skew-invariant.

2.2. Pinczon bracket.

It is defined on cyclic forms.

Definition 2.1. Let Ω be a $(k+1)$ -linear form, Ω is a cyclic form on $V[1]$, if it satisfies:

$$\Omega(x_{k+1}, x_1, \dots, x_k) = \Omega(x_1, \dots, x_{k+1}).$$

Denote $\mathcal{C}(V)$ the space of cyclic forms on $V[1]$.

Let $\sigma \in \mathfrak{S}_k$ and Ω k -linear. Put $(\Omega^\sigma)(x_1, \dots, x_k) = \Omega(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)})$. Denote $Cycl$ the subgroup of \mathfrak{S}_k generated by the cycle $(1, 2, \dots, k)$. Put

$$\Omega^{Cycl} = \sum_{\tau \in Cycl} \Omega^\tau, \quad \text{and} \quad A \odot B = (A \otimes B)^{Cycl}.$$

It is easy to prove that the so defined cyclic product is commutative, but non associative.

Definition 2.2. A Pinczon bracket $\{ , \}$ on the space $\mathcal{C}(V)$ of cyclic multilinear forms on a graded space $V[1]$ is a bilinear map such that

1. If \mathcal{C}^k is the space of k -linear cyclic forms, $\{\mathcal{C}^{k+1}, \mathcal{C}^{k'+1}\} \subset \mathcal{C}^{k+k'}$,
2. $\mathcal{C}(V)$, equipped with $\{ , \}$ is a graded Lie algebra, with center \mathcal{C}^0 ,
3. for any linear form α , $\{\alpha, \}$ is a derivation: if β_1, \dots, β_k are linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\} = \sum_j \{\alpha, \beta_j\} (\beta_1 \otimes \dots \otimes \hat{\beta}_j \otimes \dots \otimes \beta_k)^{Cycl},$$

Now,

Proposition 2.3. 1. There is a bijective map between the set \mathcal{P} of Pinczon bracket on $\mathcal{C}(V)$ and the set \mathcal{B} of degree 0, symmetric, non degenerated bilinear forms b on V ,
 2. Let b be in \mathcal{B} , and (e_i) a basis for V , if (e'_i) is the basis defined by $b(e_i, e'_j) = \delta_{ij}$, then the Pinczon bracket associated to b is:

$$\{\Omega, \Omega'\} = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega'.$$

Proof. Let $\{ , \}$ be a Pinczon bracket on $\mathcal{C}(V)$, then $\{\mathcal{C}^0, \mathcal{C}(V)\} = 0$, and $\{\mathcal{C}^1, \mathcal{C}^1\} \subset \mathbb{K}$. The bracket defines a degree 0 bilinear, antisymmetric form B^* on $\mathcal{C}^1 = (V[1])^*$. As above this defines a symmetric bilinear form b^* on the space $V^* = (V[1])^{*}[1]$.

Suppose that for some α in \mathcal{C}^1 , $B^*(\alpha, \mathcal{C}^1) = \{\alpha, \mathcal{C}^1\} = 0$. Since $\{\alpha, \cdot\}$ is a derivation, $\{\alpha, \mathcal{C}^k\} = 0$, and α is a central element. Thus $\alpha = 0$, b^* is non degenerated and allows to identify V^* and V and b^* to a non degenerated bilinear symmetric form b on V .

Let B^* be a skewsymmetric bilinear form on $(V[1])^*$. For any basis (e_i) of $V[1]$ there are vectors e'_i such that:

$$B^*(\alpha, \beta) = \sum_i \iota_{e_i} \alpha \otimes \iota_{e'_i} \beta = \sum_i \alpha(e_i) \beta(e'_i) = \sum_i \iota_{e_i} \alpha \odot \iota_{e'_i} \beta.$$

Coming back to V^* this means, if all the objects are homogeneous,

$$b^*(\alpha, \beta) = \sum_i (-1)^{|\beta|} \alpha(e_i) \beta(e'_i).$$

Identify V^* to V by defining, for any γ in V^* , the vector x_γ in V such that $\alpha(x_\gamma) = b^*(\alpha, \gamma)$, for any α in V^* . Then $e'_i = x_{\epsilon_i}$, where (ϵ_i) is the dual basis of (e_i) . Therefore (e'_i) is a basis for V , and $b(e'_j, e_i) = \delta_{ij}$.

Consider now the bracket:

$$\{\Omega, \Omega'\}_P = \sum_i \iota_{e_i} \Omega \odot \iota_{e'_i} \Omega',$$

where Ω is a $k+1$ -linear cyclic form and Ω' a $k'+1$ -linear one. In a following section, we shall prove this bracket defines a graded Lie algebra structure on $\mathcal{C}(V)$. It is clear that, for any α and β_j linear,

$$\{\alpha, (\beta_1 \otimes \dots \otimes \beta_k)^{Cycl}\}_P = \sum_j \{\alpha, \beta_j\}_P (\beta_1 \otimes \dots \hat{\beta}_j \dots \otimes \beta_k)^{Cycl}.$$

Moreover the center of the Lie algebra $(\mathcal{C}(V), \{\cdot, \cdot\}_P)$ is \mathcal{C}^0 . In other word, $\{\cdot, \cdot\}_P$ is a Pinczon bracket.

If $k + k' \leq 0$, $\{\Omega, \Omega'\} = \{\Omega, \Omega'\}_P$. Suppose by induction this relation holds for $k + k' < N$ and consider Ω and Ω' such that $k + k' = N$. For any i ,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = -\{\Omega, \{\Omega', \epsilon_i\}_P\}_P - \{\Omega', \{\epsilon_i, \Omega\}_P\}_P = \{\epsilon_i, \{\Omega, \Omega'\}_P\}_P = \iota_{e'_i} \{\Omega, \Omega'\}_P$$

On the other hand, if $\{\Omega, \Omega'\} = \sum_\beta (\beta_1 \otimes \dots \otimes \beta_{k+k'})^{Cycl}$,

$$\{\epsilon_i, \{\Omega, \Omega'\}\} = \sum_{\beta, j} \beta_j(e'_i) (\beta_1 \otimes \dots \hat{\beta}_j \dots \otimes \beta_{k+k'})^{Cycl} = \iota_{e'_i} \{\Omega, \Omega'\}.$$

This proves the existence and unicity, and gives the form of the Pinczon bracket associated to the symmetric, non degenerated bilinear form b on V .

□

3. CODIFFERENTIAL

3.1. General construction.

The deconcatenation Δ is a natural comultiplication in the tensor algebra $\bigotimes^+ V[1]$:

$$\Delta(x_1 \otimes \dots \otimes x_k) = \sum_{r=1}^{k-1} (x_1 \otimes \dots \otimes x_r) \bigotimes (x_{r+1} \otimes \dots \otimes x_k).$$

The space \mathcal{D} of coderivations of Δ is a natural graded Lie algebra for the commutator. It is well known (see for instance [K, LM]) that any multilinear mapping Q can be extended in an unique way into a coderivation D_Q of Δ , and conversely any coderivation D has an unique form $D = \sum_k D_{Q_k}$. Thus the space of multilinear mappings is a graded Lie algebra for the bracket $[Q, Q'] = Q \circ Q' - Q' \circ Q$, where, if $x_{[a,b]} = x_a \otimes x_{a+1} \otimes \dots \otimes x_b$,

$$Q \circ Q'(x_{[1,k+k'-1]}) = \sum_{r=0}^{k-1} Q(x_{[1,r]}, Q'(x_{[r+1,r+k']}, x_{[r+k'+1,k+k'-1]}).$$

3.2. Relation with the Pinczon bracket.

Consider a vector space V equipped with a symmetric, non degenerated bilinear form b , with degree 0, denote B the associated form on $V[1]$. For any linear map $Q : V[1]^k \rightarrow V[1]$, define the $(k+1)$ -linear form:

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}),$$

and let us say that Q is B -quadratic if Ω_Q is cyclic. Denote \mathcal{D}_B the space of B -quadratic multilinear maps.

The fundamental examples of cyclic maps are mappings associated to a Lie bracket or an associative multiplication on V . More precisely, if $(x, y) \mapsto q(x, y)$ is any internal law, with degree 0, and $Q(x, y) = (-1)^x q(x, y)$, then Ω_Q is cyclic if and only if:

$$b(q(x, y), z) = b(x, q(y, z)),$$

if and only if (V, q, b) is a quadratic algebra.

Remark that if q is a Lie bracket, then Ω_Q is symmetric. Now, if q is a commutative (and associative) product, then Ω_Q is vanishing on the image of the shuffle product on the 2 first variables:

$$sh_{(1,1)}(x_1 \otimes x_2) = (x_1 \otimes x_2) + (x_2 \otimes x_1).$$

Proposition 3.1. *The space \mathcal{D}_B of B -quadratic maps Q is a Lie subalgebra of \mathcal{D} .*

The space $\mathcal{C}(V)$ of cyclic forms, equipped with the Pinczon bracket is a graded Lie algebra, isomorphic to \mathcal{D}_B .

Proof. For any sequence $I = \{i_1, \dots, i_k\}$ of indices, denote x_I the tensor $x_{i_1} \otimes \dots \otimes x_{i_k}$.

Suppose Q is k -linear, Q' k' -linear, both B -quadratic. Thus:

$$\begin{aligned} B(Q(x_{[1,k]}), Q'(x_{[k,k+k']})) &= -B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \\ &= B(Q(Q'(x_{[k,k+k']}), x_{[1,k-1]}), x_k). \end{aligned}$$

Therefore:

$$\begin{aligned} B(Q \circ Q'(x_{[2,k+k']}), x_1) &= \sum_{1 \leq r} B(Q(x_{[2,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k']}), x_1) \\ &= \sum_{1 \leq r < k} B(Q(x_{[1,r]}, Q'(x_{[r,r+k']}), x_{[r+k',k+k'-1]}), x_{k+k'}) - B(Q'(Q(x_{[1,k]}), x_{[k,k+k'-1]}), x_{k+k'}) \end{aligned}$$

Or

$$B([Q, Q'](x_{[2,k+k']}), x_1) = B([Q, Q'](x_{[1,k+k'-1]}), x_{k+k'})$$

This means $[Q, Q']$ is B -quadratic, \mathcal{D}_B is a Lie subalgebra of \mathcal{D} .

Now Ω_Q (resp. $\Omega_{Q'}$) is a $k+1$ -linear (resp. $k'+1$ -linear) cyclic form, and

$$\begin{aligned} \{\Omega_Q, \Omega_{Q'}\}(x_1, \dots, x_{k+k'}) &= \left(\sum_i \iota_{e_i} \Omega_Q \otimes \iota_{e'_i} \Omega_{Q'} \right)^{Cycl} (x_1, \dots, x_{k+k'}) \\ &= \sum_{\sigma \in Cycl} \sum_i B(Q(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(k)}), e_i) B(Q'(x_{\sigma^{-1}(k+1)}, \dots, x_{\sigma^{-1}(k+k')}), e'_i) \\ &= \sum_{\sigma \in Cycl} B(Q(x_{\sigma^{-1}([1,k])}, Q'(x_{\sigma^{-1}([k+1,k+k'])})). \end{aligned}$$

Consider a term in this sum, such that $k+k'$ belongs to $\sigma^{-1}([1, k])$. This term is:

$$B(Q(x_{[r+k'+1,k+k']}, x_{[1,r]}), Q'(x_{[r+1,r+k']})) = B(Q(x_{[1,r]}, Q'(x_{[r+1,r+k']}), x_{[r+k'+1,k+k'-1]}), x_{k+k'}).$$

The sum of all these terms is just $B((Q \circ Q')(x_{[1,k+k'-1]}), x_{k+k'})$.

Similarly, if $k+k'$ is in $\sigma^{-1}([k+1, k+k'])$, we get:

$$B(Q(x_{[r+1,r+k']}, Q'(x_{[r+k+1,k+k']}, x_{[1,r]})) = -B(Q'(x_{[1,r]}, Q(x_{[r+1,r+k']}, x_{[r+k+1,k+k'-1]}), x_{k+k'}),$$

and the corresponding sum is $-B((Q' \circ Q)(x_{[1,k+k'-1]}), x_{k+k'})$.

This proves:

$$\{\Omega_Q, \Omega_{Q'}\} = \Omega_{[Q, Q']},$$

and the proposition, since $Q \mapsto \Omega_Q$ is bijective. □

Let us now study in a more detailed way different cases, when Q corresponds to an associative, or a commutative or a Lie, or a pre-Lie structure, or to an up to homotopy such structure.

4. ASSOCIATIVE PINCZON ALGEBRAS

4.1. Associative quadratic algebras.

Suppose now q is a degree 0 associative product, and b is invariant, then the associated coderivation Q of Δ , with degree 1 is the Bar resolution of the associative algebra (V, q) . The associativity of q is equivalent to the relation $[Q, Q] = 0$.

More generally, a structure of A_∞ algebra (or associative algebra up to homotopy) on the space V is a degree 1 coderivation Q of Δ on $\otimes^+ V[1]$, such that $[Q, Q] = 0$. With this last relation, the Pinczon coboundary $d_P : \Lambda \mapsto \{\Omega_Q, \Lambda\}$ is a degree 1 differential on the (graded) Lie algebra $\mathcal{C}(V)$. The corresponding cohomology is the Pinczon cohomology of cyclic forms. Write also $\Omega_{d_P Q} = d_P \Omega_Q$.

Definition 4.1. *An associative Pinczon algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 form $\Omega \in \mathcal{C}(V)$, such that $\{\Omega, \Omega\} = 0$.*

If Ω is trilinear, then an associative Pinczon algebra is simply a quadratic associative algebra (V, b, q) , where b is the symmetric non degenerated form coming from the Pinczon bracket, and q the bilinear mapping associated to Q such that $\Omega = \Omega_Q$.

Proposition 4.2. *Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be an associative Pinczon algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : \otimes^k V[1] \rightarrow V[1]$. Then Q is a structure of A_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.*

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of A_∞ algebra on V defines a unique structure of associative Pinczon algebra on V .

4.2. Bimodules and Hochschild cohomology.

Suppose (V, q) is an associative algebra and M a bi-module. Then the Hochschild cohomology with value in M is a part of the Pinczon cohomology of a natural Pinczon algebra.

Consider first the semidirect product $W = V \rtimes M$, that is the vector space $V \times M$, equipped with the associative product $q_W((x, a), (y, b)) = (q(x, y), (x \cdot b + a \cdot y))$. The dual $W^* = V^* \times M^*$ is now a (W, q_W) -bimodule, with:

$$((x, a) \cdot f)(z, c) = f((z, c)(x, a)), \quad (f \cdot (x, a))(z, c) = f((x, a)(z, c)),$$

or if $f = (g, h) \in V^* \times M^*$, $(x, a) \cdot (g, h) = (x \cdot g + a \cdot h, x \cdot h)$, $(g, h) \cdot (x, a) = (g \cdot x + h \cdot a, h \cdot x)$.

This defines a structure of algebra on the space $\tilde{V} = W \times W^*$, namely:

$$\tilde{q}((x, a, g, h), (x', a', g', h')) = (xx', xa' + ax', x \cdot g' + g \cdot x' + a \cdot h' + h \cdot a', x \cdot h' + h \cdot x'),$$

and a non degenerated symmetric bilinear form \tilde{b} ,

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

Now $(\tilde{V}, \tilde{b}, \tilde{q})$ is a quadratic associative algebra, it is the double semi-direct product of (V, q) by the bimodule M . As usual, \tilde{Q} is associated to \tilde{q} , after a shifting of degree.

Let now $c : V^k \rightarrow M$, k -linear, with degree $|c| = 2 - k$, and identify C with $\tilde{C} : \tilde{V}[1]^k \rightarrow \tilde{V}[1]$, by putting:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, g_k, h_k)) = (0, C(x_1, \dots, x_k), \sum_j C_j(x_1, \dots, h_j, \dots, x_k), 0),$$

where $C_j(x_1, \dots, h_j, \dots, x_k) = h_j(C(x_{j+1}, \dots, x_k, \cdot, x_1, \dots, x_{j-1})) \in V^*$. A direct computation shows that \tilde{C} is \tilde{B} -quadratic. A direct computation gives:

$$d_P \tilde{Q} = [\tilde{Q}, \tilde{C}] = \widetilde{[Q, C]} = \widetilde{(d_H c)[1]},$$

where d_H is the Hochschild coboundary operator on the bimodule M .

Proposition 4.3.

Let (V, q) be an associative algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear mapping c from V^k into M , with degree $2 - k$, the cyclic form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Hochschild cohomology for the (V, q) bimodule M and the Pinczon cohomology of cyclic forms $\mathcal{C}(\tilde{V})$ on \tilde{V} .

5. COMMUTATIVE PINCZON ALGEBRAS

5.1. Commutative quadratic algebras.

Consider a quadratic associative algebra (V, b, q) , but suppose now q is commutative. Consider, as above the corresponding coderivation Q . It is now anticommutative, with degree 1, and seen as a map $\underline{\otimes}^2 V[1] \rightarrow V[1]$, where $\underline{\otimes}^2 V[1]$ is the quotient of $\otimes^2 V[1]$ by the 1, 1 shuffle product $sh_{1,1}(x_1, x_2) = x_1 \otimes x_2 + x_2 \otimes x_1$.

Recall that a p, q shuffle σ is a permutation $\sigma \in \mathfrak{S}_{p+q}$ such that $\sigma(1) < \dots < \sigma(p)$ and $\sigma(p+1) < \dots < \sigma(p+q)$. Denote $Sh(p, q)$ the set of all such shuffles. Then the p, q shuffle product on $\otimes^+ V[1]$ is

$$sh_{p,q}(x_{[1,p]}, x_{[p+1,p+q]}) = \sum_{\sigma \in Sh(p,q)} x_{\sigma^{-1}(1)} \otimes \dots \otimes x_{\sigma^{-1}(p+q)}.$$

Denote $\underline{\otimes}^n(V[1])$ the quotient of $\otimes^n V[1]$ by the sum of all the image of the maps $sh_{p,n-p}$ ($0 < p < n$), and $\underline{x}_{[1,n]} = x_1 \underline{\otimes} \dots \underline{\otimes} x_n$ the class of $x_1 \otimes \dots \otimes x_n$, where the x_i

belong to $V[1]$. Finally, $\underline{\otimes}^+(V[1])$ is the sum of all $\underline{\otimes}^n(V[1])$ ($n > 0$).

On $\underline{\otimes}^+V[1]$, there is a Lie cobracket δ , defined by

$$\delta(\underline{x}_{[1,n]}) = \sum_{j=1}^{n-1} \underline{x}_{[1,j]} \otimes \underline{x}_{[j+1,n]} - \underline{x}_{[j+1,n]} \otimes \underline{x}_{[1,j]}.$$

In fact δ is well defined on the quotient and any coderivation Q of δ is characterized by its Taylor expansion $Q = \sum_k Q_k$ where each Q_k is a linear map from $\underline{\otimes}^k V[1]$ into $V[1]$ (see for instance [AAC1, BGHHW]).

Definition 5.1. *A structure of C_∞ algebra, or up to homotopy commutative algebra, on V is a degree 1 coderivation Q of δ , on $\underline{\otimes}^+V[1]$, such that $[Q, Q] = 0$.*

Associated to the notion of vanishing on shuffle products mapping Q , there is a notion of vanishing on shuffle product forms Ω .

Definition 5.2. *A $(k+1)$ -linear cyclic form Ω on the vector space $V[1]$ is vanishing on shuffle product if, for any $y, (x_1, \dots, x_k) \mapsto \Omega(x_1, \dots, x_k, y)$ is vanishing on shuffle product. Denote $\mathcal{C}_{vsp}(V)$ the space of cyclic, vanishing on shuffle product multilinear forms on $V[1]$.*

Proposition 5.3. *Suppose $\{ , \}$ is a Pinczon bracket on the space $\mathcal{C}(V)$ of cyclic multilinear forms on $V[1]$. Then $\mathcal{C}_{vsp}(V)$ is a Lie subalgebra of $(\mathcal{C}(V), \{ , \})$.*

Proof. In fact, the Pinczon bracket defines a non degenerate form b on V , thus B on $V[1]$, any form Ω can be written as $\Omega = \Omega_Q$, with

$$\Omega_Q(x_1, \dots, x_{k+1}) = B(Q(x_1, \dots, x_k), x_{k+1}).$$

Therefore Ω is in $\mathcal{C}_{vsp}(V)$ if and only if Q is vanishing on shuffle products. Now, in [AAC2] it is shown that if Q, Q' are vanishing on shuffle products, then $[Q, Q']$ is also vanishing on shuffle products. This proves the proposition. \square

Definition 5.4. *A commutative Pinczon algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 form $\Omega \in \mathcal{C}_{vsp}(V)$, such that $\{\Omega, \Omega\} = 0$.*

As for associative algebra, a commutative Pinczon algebra with Ω trilinear is simply a quadratic commutative algebra (V, q, b) .

Proposition 5.5. *Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be an commutative Pinczon algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : \otimes^k V[1] \rightarrow V[1]$. Then Q is a structure of C_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.*

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of C_∞ algebra on V defines an unique structure of commutative Pinczon algebra on V .

5.2. (Bi)modules and Harrison cohomology.

Any module M on a commutative algebra (V, q) is a bimodule where right and left action are coinciding. Let now (V, q) be a commutative algebra, and M a (V, q) -module. Repeat the preceding construction of the quadratic associative algebra $(\tilde{V}, \tilde{b}, \tilde{q})$. Now $(\tilde{V}, \tilde{b}, \tilde{q})$ is commutative.

As above, look now for a k -linear mapping c from V^k into M , with degree $2 - k$ and vanishing on shuffle products, denote \tilde{C} the corresponding map $\tilde{V}^k \rightarrow \tilde{V}$, with degree 1, We saw that $d_P \tilde{C} = [\tilde{Q}, \tilde{C}] = [\tilde{Q}, C] = \widetilde{d_H c[1]}$. If we restrict this to vanishing on shuffle products map c , this is the Harrison coboundary $\widetilde{d_H c[1]}$.

Proposition 5.6. *Let (V, q) be a commutative algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear mapping c from V^k into M , with degree $2 - k$, the cyclic form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Harrison cohomology for the (V, q) bimodule M and the Pinczon cohomology of cyclic forms $\mathcal{C}(\tilde{V})$ on \tilde{V} .*

6. PINCZON LIE ALGEBRAS

6.1. Quadratic Lie algebras.

Suppose now (V, q) is a (graded) Lie algebra. Then the corresponding Bar resolution consists in replacing the space $\otimes^+ V[1]$ of tensors by the subspace $S^+(V[1])$ of symmetric tensors, spanned by the symmetric products $x_1 \cdot \dots \cdot x_k = \sum_{\sigma \in \mathfrak{S}_k} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$. Then, the natural comultiplication Δ on $S^+(V[1])$ is:

$$\Delta(x_1 \cdot \dots \cdot x_k) = \sum_{\substack{I \sqcup J = [1, k] \\ 0 < \#I < k}} x_{\cdot I} \otimes x_{\cdot J},$$

where, if $I = \{i_1 < \dots < i_r\}$, $x_{\cdot I}$ means $x_{i_1} \cdot \dots \cdot x_{i_r}$.

As above, any coderivation Q of the comultiplication Δ is characterized by its Taylor coefficients $Q_k : S^k(V[1]) \rightarrow V[1]$. The bracket of two such coderivations Q ,

Q' becomes:

$$[Q, Q'](x_1 \cdot \dots \cdot x_{k+k'-1}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q(Q'(x_{.J}) \cdot x_{.I}) - \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#I = k}} Q'(Q(x_{.I}) \cdot x_{.J}),$$

Now to the Lie bracket q on V is associated a map $Q : S^2(V[1]) \rightarrow V[1]$, thus a degree 1 coderivation, still denoted Q , of Δ and the Jacobi identity for q is equivalent to the relation $[Q, Q] = 0$. The corresponding cohomology is the Chevalley cohomology.

For any k , consider $S^k(V[1])$ as a subspace of $\otimes^k V[1]$, with a projection $Sym : \otimes^k V[1] \rightarrow S^k(V[1])$:

$$Sym(x_1 \otimes \dots \otimes x_k) = x_1 \cdot \dots \cdot x_k.$$

Denote $L(\otimes^k V[1], V[1])$ the of linear maps $Q : \otimes^k V[1] \rightarrow V[1]$. By restriction to $S^k(V[1])$, Q defines a symmetric map, denote it Q^{Sym} . Then $Q^{Sym} = Q \circ Sym$, and the space $L(S^k(V[1]), V[1])$ of symmetric maps is a quotient of $L(\otimes^k V[1], V[1])$.

Similarly, the restriction of $\Omega \in \mathcal{C}^{k+1}$ is a symmetric form, denote it Ω^{Sym} : $\Omega^{Sym} = \Omega \circ Sym$. as above, the space $\mathcal{C}|_S(V)$ of the restrictions of cyclic multilinear forms to $S^+(V[1])$ is a quotient of $\mathcal{C}(V)$.

Suppose there is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, then any Ω can be written $\Omega = \Omega_Q$, with Q k -linear and B -quadratic. But, any element σ in \mathfrak{S}_{k+1} can be written in an unique way as a product $\tau \circ \rho$ where τ is in \mathfrak{S}_k , viewed as a subgroup of \mathfrak{S}_{k+1} and ρ in $Cycl$. Therefore, with our notation,

$$\Omega^{Sym} = (\Omega_Q)^{Sym} = (\Omega_{Q^{Sym}})^{Cycl} = (k+1)\Omega_{Q^{Sym}}.$$

Proposition 6.1. *The bracket defined on $L(\otimes^+ V[1], V[1])$ by the commutator of coderivations in $(\otimes^+ V[1], \Delta)$, induces a well defined bracket on $L(S^+(V[1]), V[1])$, this bracket is the above commutator of coderivations in $(S^+(V[1]), \Delta)$:*

$$[Q^{Sym}, (Q')^{Sym}] = [Q, Q']^{Sym}.$$

Any Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$ induces a well defined bracket on the quotient $\mathcal{C}|_S(V)$, this bracket denoted $\{ , \}$ is:

$$\{\Omega^{Sym}, (\Omega')^{Sym}\} = (\{\Omega, \Omega'\})^{Sym} = \frac{k+k'}{(k+1)!(k'+1)!} \sum_i \iota_{e_i}(\Omega^{Sym}) \cdot \iota_{e'_i}((\Omega')^{Sym}).$$

Proof. The first assertion is a simple computation:

$$(Q \circ Q')(x_{[1, k+k'-1]}) = \sum_{r=1}^{k-1} Q(x_{[1, r-1]} \otimes Q'(x_{[r, r+k'-1]}) \otimes x_{[r+k', k+k'-1]}).$$

Then,

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{\substack{r \\ \sigma \in \mathfrak{S}_{k+k'-1}}} Q(x_{\sigma^{-1}([1, r-1])}, Q'(x_{\sigma^{-1}([r, r+k'-1])}), x_{\sigma^{-1}([r+k', k+k'-1])})$$

Fix r and σ , put $J = \sigma^{-1}([r, r+k'-1]) = \{j_1 < \dots < j_{k'}\}$, and $j_{\tau^{-1}(t)} = \sigma^{-1}(r+t-1)$ ($1 \leq t \leq k'$). This define $\sigma \in \mathfrak{S}_{k'}$. Similarly, put $I = [1, k+k'-1] \setminus J$, $\{0\} \cup I = \{i_1 < \dots < i_k\}$ and define $\rho \in \mathfrak{S}_k$ by:

$$i_{\rho^{-1}(t)} = \sigma^{-1}(t) \quad (1 \leq t \leq r-1), \quad i_{\rho^{-1}(r)} = 0, \quad i_{\rho^{-1}(t)} = \sigma^{-1}(t-1) \quad (r+1 \leq t \leq k).$$

The correspondence $(r, \sigma) \mapsto (r, \tau, \rho)$ is one-to-one and, suming up, we get:

$$(Q \circ Q')^{Sym}(x_{[1, k+k'-1]}) = \sum_{\substack{I \sqcup J = [1, k+k'-1] \\ \#J = k'}} Q^{Sym}((Q')^{Sym}(x_{.J}) \cdot x_{.I}).$$

The commutator of coderivation of $(S^+(V[1]), \Delta)$ is the quotient bracket:

$$[Q, Q']^{Sym} = [Q^{Sym}, (Q')^{Sym}].$$

Let now $\{, \}$ be a Pinczon bracket on the space $\mathcal{C}(V)$. Thus any cyclic form Ω is written as Ω_Q . Then:

$$\{\Omega, \Omega'\}^{Sym} = \{\Omega_Q, \Omega_{Q'}\}^{Sym} = \Omega_{[Q, Q']}^{Sym} = (k+k')\Omega_{[Q, Q']^{Sym}} = (k+k')\Omega_{[Q^{Sym}, (Q')^{Sym}]}.$$

The last bracket is the commutator of coderivations in $S^+(V[1])$, thus

$$\{\Omega, \Omega'\}^{Sym}(x_{[1, k+k']}) = (k+k') \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})).$$

On the other hand,

$$\iota_{e_i} \left(\Omega_Q^{Sym} \right) (x_{[1, k]}) = (k+1) B(Q^{Sym}(x_{[1, k]}), e_i) = (k+1) (\iota_{e_i} \Omega)^{Sym}(x_{[1, k]}).$$

Therefore

$$\begin{aligned} \sum_i \iota_{e_i} \left(\Omega_Q^{Sym} \right) \cdot \iota_{e'_i} \left(\Omega_{Q'}^{Sym} \right) (x_{[1, k+k']}) &= \sum_i \left(\iota_{e_i} \left(\Omega_Q^{Sym} \right) \otimes \iota_{e'_i} \left(\Omega_{Q'}^{Sym} \right) \right)^{Sym} (x_{[1, k+k']}) \\ &= (k+1)!(k'+1)! \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#J = k'}} B(Q^{Sym}(x_{.I}), (Q')^{Sym}(x_{.J})) \end{aligned}$$

This achieves the proof. \square

Explicitely, if the forms Ω and Ω' are symmetric, $\Omega^{Sym} = (k+1)!\Omega$ and the Pinczon bracket on the quotient becomes the bracket defined in [PU, DPU]:

$$\{\Omega, \Omega'\} = (k+k') \sum_i \iota_{e_i} \Omega \cdot \iota_{e'_i} \Omega'.$$

6.2. Quadratic L_∞ algebras.

As above,

Definition 6.2. A Pinczon Lie algebra $(\mathcal{C}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, and a degree 3 symmetric form $\Omega \in \mathcal{C}(V)^{Sym}$, such that $\{\Omega, \Omega\} = 0$.

Since a structure of L_∞ algebra (or Lie algebra up to homotopy) on V is a degree 1 coderivation Q of $(S^+(V[1]), \Delta)$, such that the commutator $[Q, Q]$ vanishes. With the preceding computations, a Pinczon Lie algebra is in fact a quadratic L_∞ algebra.

Proposition 6.3. Let $(\mathcal{C}(V), \{ , \}, \Omega)$ be a Pinczon Lie algebra. Write $\Omega = \Omega_Q$, $Q = \sum_k Q_k$, with $Q_k : S^k(V[1]) \rightarrow V[1]$. Then Q is a structure of L_∞ algebra on V , and each Q_k is B -quadratic for the bilinear form B coming from the bracket.

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of L_∞ algebra on V defines an unique structure of commutative Pinczon algebra on V .

6.3. Modules and Chevalley cohomology.

Let (V, q) be a Lie algebra, and M a (V, q) -module. To refine the corresponding Chevalley coboundary operator d_{Ch} , build, as above, the double semidirect product of V by M .

First, consider the semidirect product $W = V \rtimes M$, that is the vector space $V \oplus M$, equipped with the Lie bracket $q_W((x, a), (y, b)) = ([x, y], x \cdot b - y \cdot a)$; its dual W^* is also a (W, q_W) -module.

Then $\tilde{V} = W \times V \times W^*$ is a quadratic Lie algebra for the bracket

$$\tilde{q}((x, a, g, h), (x', a', g', h')) = ([x, x'], x \cdot a' - x' \cdot a, x \cdot g' - x' \cdot g + a \cdot h' - a' \cdot h, x \cdot h' - x' \cdot h),$$

and the non degenerated symmetric bilinear form \tilde{b}

$$\tilde{b}((x, a, g, h), (x', a', g', h')) = g(x') + h(a') + g'(x) + h'(a).$$

A direct computation (see [BB, MR]) shows that $(\tilde{V}, \tilde{b}, \tilde{q})$ is a quadratic Lie algebra, the double semidirect product of (V, q) by its module M .

Look now for a skew-symmetric k -linear mapping c from V^k into M , with degree $|c| = 2 - k$. Associate to it the mapping:

$$\tilde{C}((x_1, a_1, g_1, h_1), \dots, (x_k, a_k, f_k, g_k)) = (0, C(x_1, \dots, x_k), \sum_{j=1}^k C_j(x_1, \dots, h_j, \dots, x_k), 0).$$

Clearly, \tilde{C} is totally symmetric from $\tilde{V}[1]^k$ into $\tilde{V}[1]$. More precisely $\widetilde{C^{Sym}} = \tilde{C}^{Sym}$.

Then,

$$[\tilde{Q}, \tilde{C}]^{Sym} = \widetilde{[Q, C]^{Sym}} = [\widetilde{Q}, \widetilde{C}]^{Sym} = \widetilde{d_{Ch}c[1]}.$$

Now, if d_P is the Pinczon coboundary operator, defined by:

$$\{\Omega_{\tilde{Q}}, \Omega_{\tilde{C}}\} = \Omega_{d_P \tilde{C}},$$

this can be written $d_P \tilde{C} = (2 + k) \widetilde{d_{Ch}c[1]}$.

Proposition 6.4. *Let (V, q) be a Lie algebra, and $c \mapsto \Omega_{\tilde{C}}$ the map associating to any multilinear skewsymmetric mapping c from V^k into M , with degree $2 - k$, the symmetric form $\Omega_{\tilde{C}}$. Then this map is a complex morphism between the Chevalley cohomology for the (V, q) module M and the Pinczon cohomology of symmetric forms $\mathcal{C}(\tilde{V})|_S$ on \tilde{V} .*

7. PINCZON PRE-LIE ALGEBRAS

7.1. Quadratic pre-Lie algebras.

A left pre-Lie algebra (V, q) is a (graded) vector space with a product q such that:

$$q(x, q(y, z)) - q(q(x, y), z) = q(y, q(x, z)) - q(q(y, x), z).$$

Then the bracket $[x, y] = q(x, y) - q(y, x)$ is a Lie bracket. Remark that any associative algebra (V, q) is a pre-Lie algebra.

A vector space M is a left (V, q) -module for the linear map $x \otimes a \mapsto x \cdot a$, if

$$q(x, y) \cdot a - x \cdot (y \cdot a) = q(y, x) \cdot a - y \cdot (x \cdot a) \quad (a \in M, x, y \in V).$$

A left module is a bi-module, if there is a linear map $a \otimes x \mapsto a * x$ such that:

$$(a * x) * y - a * q(x, y) = (x \cdot a) * y - x \cdot (a * y) \quad (a \in M, x, y \in V).$$

Then a direct computation proves

Lemma 7.1. *Let (V, q) a left pre-Lie algebra, then*

1. *The dual V^* of V is a left (V, q) -module, for: $(x \cdot f)(y) = -f(q(x, y))$.*
2. *Let M be a (V, q) bi-module then $W = V \rtimes M = (V \oplus M, q_W)$, where:*

$$q_W(x + a, y + b) = q(x, y) + x \cdot b + a * y$$

is a left pre-Lie algebra, the semi-direct product of V by M .

Chapoton and Livernet define in [CL] the notion of pre- L_∞ -algebra. If V is a graded vector space, we consider the space $S(V[1]) \otimes V([1])$, generated by the tensors

$$x_{[1,k]} \otimes y = x_1 \cdot \dots \cdot x_k \otimes y.$$

Put $P^k = S^k(V[1]) \otimes V([1])$, and $P = \sum_{k \geq 0} P^k$. On P , the coproduct Δ is defined by $\Delta(1 \otimes y) = 0$ and:

$$\Delta(x_{[1,k]} \otimes y) = \sum_{\substack{1 \leq j \leq k \\ I \sqcup J = [1,k] \setminus \{j\}}} (x_I \otimes x_j) \otimes (x_J \otimes y).$$

A linear map $Q : P^k \rightarrow P^0$ extends to a coderivation, still denoted Q by:

$$Q(x_{[1,n]} \otimes y) = \sum_{\substack{I \sqcup J = [1,n] \\ \#J=k}} x_I \otimes Q(x_J \otimes y) + \sum_{\substack{1 \leq j \leq n \\ I \sqcup J = [1,n] \setminus \{j\} \\ \#J=k}} x_I \cdot Q(x_J \otimes x_j) \otimes y.$$

The commutator of two coderivations is a coderivation, and a map $q : V \otimes V \rightarrow V$ is a left pre-Lie product if and only if the structure equation $[Q, Q] = 0$ holds for the corresponding map $Q : P^1 \rightarrow P^0$. Thus:

Definition 7.2. *A structure of pre- L_∞ algebra on V is a degree 1 coderivation Q of (P, Δ) , such that $[Q, Q] = 0$.*

Now, a quadratic pre-Lie algebra (V, b, q) is a pre-Lie algebra (V, q) equipped with a symmetric, non degenerated, degree 0, bilinear form b such that

$$b(q(x, y), z) + b(y, q(x, z)) = 0, \quad \text{or} \quad B(Q(x, y), z) = B(Q(x, z), y).$$

Example 7.3. *Let (V, q) be a left pre-Lie algebra, consider the pre-Lie algebra, semidirect product $W = V \rtimes V^*$, with*

$$q_W(x + f, y + g) = q(x, y) + x \cdot g = q(x, y) - g(q(x, \cdot)).$$

It is a quadratic pre-Lie algebra if we endow W by the canonical symmetric, non degenerated form $b(x + f, y + g) = f(y) + g(x)$.

Generalizing, let us say that a coderivation $Q = Q_0 + Q_1 + \dots$ of Δ is B -quadratic if, for any k ,

$$B(Q_k(x_{[1,k]} \otimes y_1), y_2) = B(Q_k(x_{[1,k]} \otimes y_2), y_1).$$

Then a direct computation gives:

Lemma 7.4. *Let Q and Q' be two B -quadratic coderivations of Δ . Then $[Q, Q']$ is B -quadratic.*

A structure of quadratic pre- L_∞ algebra on (V, b) is thus a B -quadratic coderivation Q of Δ such that $[Q, Q] = 0$.

7.2. Pinczon bracket for pre-Lie algebras.

To a $k + 1$ -linear coderivation Q , let us associate the form Ω_Q :

$$\Omega_Q(x_{[1,k]} \otimes y_1 \cdot y_2) = B(Q(x_{[1,k]} \otimes y_1), y_2) + B(Q(x_{[1,k]} \otimes y_2), y_1).$$

This form is separately symmetric in its k first variables, and its 2 last variables. Define thus \mathcal{P}_k the space of such forms, and $\mathcal{P}(V) = \sum_{k \geq 0} \mathcal{P}_k$. An element of $\mathcal{P}(V)$ is called a bi-symmetric form. Now it is possible to extend to $\mathcal{P}(V)$ the Pinczon bracket $\{ , \}$ associated to B . Indeed, a direct computation shows:

Lemma 7.5. *Let \mathfrak{g} be a Lie algebra and A a commutative algebra which is a right \mathfrak{g} -module such that, for any $x \in \mathfrak{g}$, $f \mapsto f \cdot x$ is a derivation of A . Then the formula:*

$$[f \otimes x, g \otimes y] = fg \otimes [x, y] + (f \cdot y)g \otimes x - f(g \cdot x) \otimes y$$

defines a Lie bracket on $A \otimes \mathfrak{g}$.

Denote $\mathcal{S}(V) = (\mathcal{C}(V))^{\text{Sym}}$ the symmetric algebra of V , it is a commutative algebra for the symmetric product \cdot . On the other hand, by construction, \mathcal{C}^2 is a Lie algebra for the Pinczon bracket $\{ , \}$, acting on \mathcal{S} through $(\Omega, \alpha) \mapsto \{\Omega, \alpha\}$. Then the properties of the Pinczon bracket assure that \mathcal{S} is a $(\mathcal{S}^2, \{ , \})$ -module and the action is a derivation of \mathcal{S} . Finally remark that $\mathcal{P}(V) = \mathcal{S}(V) \otimes \mathcal{C}^2$, therefore:

Corollary 7.6. *Let (V, b) be a vector space with a symmetric, non degenerated bilinear form, then the space $\mathcal{P}(V)$ of bi-symmetric forms on V is a Lie algebra for the Pinczon bracket:*

$$\{\Omega \otimes \alpha, \Omega' \otimes \alpha'\} = \Omega \cdot \{\alpha, \Omega'\} \otimes \alpha' + \Omega' \cdot \{\Omega, \alpha'\} \otimes \alpha + \Omega \cdot \Omega' \otimes \{\alpha, \alpha'\}.$$

This bracket is related to the commutator of B -preserving coderivations, since:

Proposition 7.7. *Suppose b is a symmetric non degenerated form on V , and Q, Q' two B -quadratic coderivations of Δ . Consider the forms $\Omega_Q, \Omega_{Q'}$ in $\mathcal{P}(V)$, then*

$$\{\Omega_Q, \Omega_{Q'}\} = 2\Omega_{[Q, Q']}.$$

Proof. Suppose $\Omega_Q = \beta \otimes \alpha \in \mathcal{P}^k$, and $\Omega_{Q'} = \beta' \otimes \alpha' \in \mathcal{P}^{k'}$, then

$$\begin{aligned} \beta \cdot \beta' \otimes \{\alpha, \alpha'\}(x_{[1, k+k']}, y_{[1, 2]}) &= \\ &= \sum_{\substack{i, I \sqcup J = [1, k+k'] \\ \#I=k}} (\Omega_Q(x_I \otimes e_i \cdot y_1) \Omega_{Q'}(x_J \otimes e'_i \cdot y_2) + \Omega_Q(x_I \otimes e_i \cdot y_2) \Omega_{Q'}(x_J \otimes e'_i \cdot y_1)) \\ &= 4 \sum_{\substack{I \sqcup J = [1, k+k'] \\ \#I=k}} -B(Q'(x_J \otimes Q(x_I \otimes y_1)), y_2) + B(Q(x_I \otimes Q'(x_J \otimes y_1)), y_2) \end{aligned}$$

On the other hand,

$$\begin{aligned}
\beta \cdot \{\alpha, \beta'\} \otimes \alpha'(x_{[1,k+k']}, y_{[1,2]}) &= \\
&= \sum_{\substack{j, I \sqcup J = [1,k+k'] \setminus \{j\} \\ \#I=k}} \sum_i \Omega_Q(x_I \otimes e_i \cdot x_j) \Omega_{Q'}(e'_i \cdot x_J \otimes y_1 \cdot y_2) \\
&= -4 \sum_{\substack{j, I \sqcup J = [1,k+k'] \setminus \{j\} \\ \#I=k}} B(Q'(Q(x_I \otimes x_j) \cdot x_J \otimes y_1), y_2).
\end{aligned}$$

And similarly for the last term $\{\beta, \alpha'\} \cdot \beta' \otimes \alpha$ in the Pinczon bracket. Suming up, we get:

$$\begin{aligned}
\{\Omega_Q, \Omega_{Q'}\}(x_{[1,k+k']} \otimes y_{[1,2]}) &\stackrel{\deg}{=} 2B([Q, Q'](x_{[1,k+k']} \otimes y_1), y_2) + 2B([Q, Q'](x_{[1,k+k']} \otimes y_2), y_1) \\
&\stackrel{\deg}{=} 2\Omega_{[Q, Q']}(x_{[1,k+k']} \otimes y_{[1,2]}).
\end{aligned}$$

□

As for the other sort of algebras,

Definition 7.8. A Pinczon pre-Lie algebra $(\mathcal{P}(V), \{ , \}, \Omega)$ is a Pinczon bracket $\{ , \}$ on $\mathcal{C}(V)$, extended to $\mathcal{P}(V)$, and a degree 3 bi-symmetric form $\Omega \in \mathcal{P}(V)$, such that $\{\Omega, \Omega\} = 0$.

Since a structure of pre- L_∞ algebra on V is a degree 1 coderivation Q of (P, Δ) , such that the commutator $[Q, Q]$ vanishes. With the preceding computations, a Pinczon pre-Lie algebra is in fact a quadratic pre- L_∞ algebra.

Proposition 7.9. Let $(\mathcal{P}(V), \{ , \}, \Omega)$ be a Pinczon pre-Lie algebra. Write $\Omega = \Omega_Q$, and $Q = \sum_k Q_k$, with $Q_k : P^k \rightarrow P^0$. Then Q is a pre- L_∞ algebra structure on V , and each Q_k is B -quadratic.

Conversely, if (V, b) is a vector space with a non degenerated symmetric bilinear form, any B -quadratic structure Q of pre- L_∞ algebra on V defines an unique structure of Pinczon pre-Lie algebra on V .

7.3. Pinczon and pre-Lie algebra cohomologies.

Suppose (V, q) is a left pre-Lie algebra and let Q be the coderivation of Δ associated to q . Since $[Q, Q] = 0$, $d : C \mapsto [Q, C]$ is a coboundary operator.

Now, let M be a (V, q) bi-module. Let $W = V \rtimes M$ be the semi-direct product of V by M . Any map $c : \wedge^k V \otimes V \rightarrow M$ can be naturally extended to a map, still denoted c , from $\wedge^k W \otimes W$ to W . If C is the corresponding coderivation, the map dc , where the coderivation corresponding to $dc : \wedge^{k+1} W \otimes W \rightarrow W$ is $[Q, C]$, is the extension

of a map $d_{pLC} : \wedge^{k+1}V \otimes V \rightarrow M$. The operator d is the pre-Lie coboundary operator. In an unpublished work, Ridha Chatbouri computed explicitly this operator:

Proposition 7.10. *The cohomology of the pre-Lie algebra (V, q) is defined as follows. Let M be a (V, q) bi-module, the cohomology with value in M is given by the following operator: If $c : \wedge^k V \otimes V \rightarrow M$ is a $(k+1)$ -cochain, with degree $|c|$, then dc is explicitly:*

$$\begin{aligned} (-1)^{|c|} d_{pLC}(x_0 \wedge \dots \wedge x_k \otimes y) = \\ \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes x_i) * y - \sum_{i=0}^k (-1)^i c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes q(x_i, y)) \\ + \sum_{i < j} (-1)^{i+j} c([x_i, x_j] \wedge x_0 \wedge \dots \hat{i} \hat{j} \dots \wedge x_k \otimes y) + \sum_{i=0}^k (-1)^i x_i \cdot c(x_0 \wedge \dots \hat{i} \dots \wedge x_k \otimes y). \end{aligned}$$

In [Dz], A. Dzhumaldil'daev defined a coboundary operator d for right pre-Lie algebra, which is the same as the operator computed in the preceding proposition, modulo the change of side for pre-Lie axioms. Then he used this operator to compute a corresponding homology. The proof of the proposition is the inverse of the Dzhumaldil'daev proof.

Remark 7.11. *A left (V, q) -module is nothing else than a $(V, [\ , \], q)$ -module. However the symmetry of a pre-Lie cochain differs of the symmetry of a Lie cochain. Thus the cocycles are not the same. For instance we consider $V = C_c^\infty(\mathbb{R})$, with $q(f, g) = fg'$. Choose $M = V$ and $f \cdot g = q(f, g)$. Then M is a left module. Put $c(f, g) = fg$. It is easy to verify it is a cocycle, but it is not skewsymmetric. In fact it is the coboundary of $f \mapsto b(f)$, with $(b(f))(t) = tf(t)$.*

More generally, if Q is a coderivation of Δ , which is a pre- L_∞ structure, then the operator $C \mapsto [Q, C]$ is a coboundary operator. Let us call the corresponding cohomology the (V, Q) cohomology. Then

Corollary 7.12. *Suppose (V, b, q) is a quadratic pre-Lie algebra or, more generally, $(\mathcal{P}(V), \{ \ , \ \}, \Omega)$ a Pinczon pre-Lie algebra. Then the operator $d : \mathcal{P}(V) \rightarrow \mathcal{P}(V)$ defined by $d\Omega = \{\Omega_Q, \Omega\}$ is a coboundary operator and the corresponding cohomology coincides with the (V, Q) -cohomology.*

8. A NATURAL EXAMPLE

Recall that the infinitesimal deformations of an associative (resp. Lie, resp. pre-Lie) algebra (V, q) are described by the corresponding second cohomology group

(Hochschild, pre-Lie or Chevalley) of V (see [G, NR]). Indeed, putting $q_t = q + tc$, with $t^2 = 0$, the associativity, Jacobi or pre-Lie relation are respectively equivalent to $[Q, C] = 0$. Therefore, if $t^2 = 0$, $(V + tV, q_t)$, is an associative, Lie or pre-Lie algebra if and only if c is respectively a Hochschild, a Chevalley or a pre-Lie cocycle.

Such a deformation is trivial if there is a linear map a such that $\varphi_t(x) = x + ta(x)$ satisfies $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$. With $t^2 = 0$, these conditions are equivalent to $c = d_H a$, resp. $c = d_{Ch} a$, $c = d_{pL} a$. If the only infinitesimal deformations are the trivial ones, that is if the second Hochschild cohomology group $H^2((V, q))$ vanishes, we say that the corresponding structure is infinitesimally rigid.

Suppose now (V, q) is an associative algebra. Therefore it is a pre-Lie algebra for the multiplication q , and a Lie algebra for the bracket $[x, y] = q(x, y) - q(y, x)$. Some of these structures can be rigid, and other can be not rigid. Let us study here the natural associative algebra $M_n(\mathbb{K})$ of $n \times n$ matrices on a characteristic zero field \mathbb{K} . Denote $\mathfrak{gl}_n(\mathbb{K})$ the corresponding Lie algebra, and (M_n, q) the pre-Lie algebra. Remark that $\mathfrak{gl}_n(\mathbb{K})$ is a direct product: $\mathfrak{sl}_n(\mathbb{K}) \oplus \mathbb{K}id$, where $\mathfrak{sl}_n(\mathbb{K})$ is the space of traceless matrices. Put $f(x) = \frac{1}{n}\text{tr}(x)id$. Recall first some well-known results (see [CH] Chap IX.7, and use Hochschild-Serre sequence and Whitehead Lemmas [J]):

Proposition 8.1. *Let M be a $M_n(\mathbb{K})$ bi-module, then $H^k(M_n(\mathbb{K}), M) = 0$ for $k > 0$. Especially, if $M = M_n(\mathbb{K})$,*

$$H^0(M_n(\mathbb{K})) = \mathbb{K}id, \quad H^k(M_n(\mathbb{K})) = 0 \quad \text{for } k > 0.$$

Let M be a $\mathfrak{gl}_n(\mathbb{K})$ -module, and $k > 0$, then:

$$H^k(\mathfrak{gl}_n(\mathbb{K}), M) \simeq H^k(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes M^{\mathfrak{gl}_n(\mathbb{K})} \oplus H^{k-1}(\mathfrak{sl}_n(\mathbb{K}), \mathbb{K}) \otimes (H^1(\mathbb{K}id, M))^{\mathfrak{gl}_n(\mathbb{K})}.$$

Especially, if $M = \mathfrak{gl}_n(\mathbb{K})$, and \tilde{f} is the class of the projection f ,

$$H^0(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}id, \quad H^1(\mathfrak{gl}_n(\mathbb{K})) = \mathbb{K}\tilde{f}, \quad H^2(\mathfrak{gl}_n(\mathbb{K})) = 0.$$

Especially, $M_n(\mathbb{K})$ and $\mathfrak{gl}_n(\mathbb{K})$ are rigid. However the pre-Lie algebra V is not infinitesimally rigid:

Lemma 8.2. *The second cohomology group $H^2(M_n, q)$ of the pre-Lie algebra (M_n, q) is not vanishing.*

Proof. Following [Dz], for any matrix a , consider $c_a(x, y) = \frac{1}{n}\text{tr}(x)[y, a]$. Then:

$$d_H c_a(x, y, z) = \frac{1}{n}(\text{tr}(y)x - \text{tr}(xy)id + \text{tr}(x)y)[z, a] = (d_H f)(x, y)[z, a].$$

Then $d_{pL} c_a(x, y, z) = d_H c_a(x, y, z) - d_H c_a(y, x, z) = 0$. But remark that $Z^2(M_n(\mathbb{K})) = B^2(M_n(\mathbb{K})) = B^2(M_n, q)$. Thus, if a is not a scalar matrix, there is z such that

$[z, a] \neq 0$, and $d_H c_a(e_{12}, e_{21}, z) = -[z, a] \neq 0$, c_a is not in $B^2(M_n, q)$. \square

Let $(x, y) \mapsto q(x, y)$ any algebraic structure on a space V and $c_k : V \otimes V \rightarrow V$ ($k \geq 1$). If for any t , $q_t = q + \sum_{k>0} t^k c_k$ endows $V \otimes \mathbb{K}[[t]]$ with the same sort of structure, we say that q_t is a formal deformation of q . If there is $\varphi_t = id + \sum_{k>0} t^k b_k$, such that $q_t(\varphi_t(x), \varphi_t(y)) = \varphi_t(q(x, y))$, we say that q_t is trivial. If there is a non trivial deformation q_t , we say that (V, q) is not rigid. If moreover $c_k = 0$ for any $k > 1$, and $c_1 \notin B^2(V, q)$, we say that $q_t = q + tc_1$ is a true deformation at order 1 of (V, q) .

Using induction on k , it is easy to prove that an infinitesimally rigid structure is rigid. Of course the existence of a true deformation at order 1 implies that (V, q) is not rigid.

Corollary 8.3. *The algebras $M_n(\mathbb{K})$, $\mathfrak{gl}_n(\mathbb{K})$ are rigid. The pre-Lie algebra (M_n, q) admits true deformations at order 1.*

Proof. The first points are the previous results. Consider now the pre-Lie algebra (M_n, q) . A direct computation shows that $q_t = q + tc_a$ is a true deformation at order 1 of (V, q) (see also [Dz]). \square

Suppose now $\mathbb{K} = \mathbb{C}$. Then $x \mapsto \text{tr}(x^2)$, and $x \mapsto (\text{tr}(x))^2$ generate the space of $\mathfrak{gl}_n(\mathbb{C})$ -invariant degree 2 polynomial functions on $\mathfrak{gl}_n(\mathbb{C})$ (see [W]). Any symmetric invariant bilinear form b on $\mathfrak{gl}_n(\mathbb{C})$ can be written:

$$b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y) = \alpha \text{tr}(x(y - f(y))) + \left(\frac{1}{n}\alpha + \beta\right) \text{tr}(x)\text{tr}(y).$$

Thus, b is non degenerate, if and only if $\alpha(\alpha + n\beta) \neq 0$.

Observe now that $(x, y) \mapsto \text{tr}(xy)$ is $M_n(\mathbb{C})$ -invariant, but $(x, y) \mapsto \text{tr}(x)\text{tr}(y)$ is not $M_n(\mathbb{C})$ -invariant. Thus the non degenerated bilinear form $(x, y) \mapsto \text{tr}(xy)$ generates the cone of $M_n(\mathbb{C})$ -invariant symmetric bilinear forms.

Similarly, if b is (M_n, q) -invariant, it is $\mathfrak{gl}_n(\mathbb{C})$ -invariant, thus $b(x, y) = \alpha \text{tr}(xy) + \beta \text{tr}(x)\text{tr}(y)$, and the invariance relation reads:

$$\alpha \text{tr}(xyz + yxz) + \beta(\text{tr}(xy)\text{tr}(z) + \text{tr}(y)\text{tr}(xz)) = 0.$$

Choosing for instance $x = y = z = e_{11}$, we get $2(\alpha + \beta) = 0$, then $x = y = z = id$ gives $2n(\alpha + n\beta) = 0$, the unique (M_n, q) -invariant symmetric bilinear form is identically zero. Summarizing,

Lemma 8.4. 1. *The associative algebra $M_n(\mathbb{C})$ is a quadratic algebra for the 1-dimensional cone of symmetric bilinear forms $\alpha \text{tr}(xy)$,*

2. The Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ is a quadratic Lie algebra for the 2-dimensional cone of symmetric bilinear forms $\alpha \operatorname{tr}(xy) + \beta \operatorname{tr}(x)\operatorname{tr}(y)$ ($\alpha(\alpha + n\beta) \neq 0$),
3. The pre-Lie algebra (M_n, q) is not a quadratic pre-Lie algebra.

Recall that in Example 7.3, we saw that the space $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$ is a quadratic pre-Lie algebra for the product and the bilinear form:

$$q_W(x_1 + x_2, y_1 + y_2) = x_1 y_1 - y_2 x_1, \quad b(x_1 + x_2, y_1 + y_2) = \operatorname{tr}(x_2 y_1 + x_1 y_2).$$

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